Box models.

Many questions in probability can be answered by considering an appropriate *box model.*

A box model is comprised of two components.

(i) A (hypothetical) box of tickets, each of which is labelled in various ways, and

(ii) A number of draws from the box.

We will imagine drawing tickets from the box in one of two ways.

(*) ***With replacement*** – after a ticket is drawn and observed, it is replaced in the box. In this case the composition of the box doesn't change from draw to draw, and the results of the different draws are *independent.*

(*) ***Without replacement*** – each ticket that is drawn from the box stays out of the box for the remaining draws. In this case, the composition of the box changes from draw to draw, and the results of the different draws are generally *not* independent.

Example 9. A box contains 50 tickets: 20 red tickets, 15 blue tickets, 10 green tickets and 5 orange tickets.

(*) If 3 tickets are drawn from the box at random ***with*** replacement, what is the probability that all three of the tickets are blue?

$\Rightarrow$ results of the draws are independent, so

$$\begin{aligned} &P(\text{1st blue and 2nd blue and 3rd blue}) \\ &\qquad = P(\text{1st blue}) \cdot P(\text{2nd blue}) \cdot P(\text{3rd blue}) \\ &\qquad = \frac{15}{50} \cdot \frac{15}{50} \cdot \frac{15}{50} = 2.7\% \end{aligned}$$

(*) If 3 tickets are drawn from the box at random ***without*** replacement, what is the probability that all three of the tickets are blue?

$\Rightarrow$ results of the draws are ***not*** independent, so

$$
\begin{aligned}
&P(\text{1st blue and 2nd blue and 3rd blue}) \\
&\quad = P((\text{1st blue and 2nd blue}) \text{ and 3rd blue}) \\
&\quad = P(\text{1st and 2nd blue}) \cdot P(\text{3rd blue}|\text{1st and 2nd blue}) \\
&\quad = \overbrace{P(\text{1st blue}) \cdot P(\text{2nd blue}|\text{1st blue})}^{P(\text{1st and 2nd blue})} \cdot P(\text{3rd blue}|\text{1st and 2nd blue}) \\
&\quad = \frac{15}{50} \cdot \frac{14}{49} \cdot \frac{13}{48} = 2.32\%
\end{aligned}
$$

Example 10. A box contains 1 red ticket, 2 blue tickets and 2 yellow tickets. If one ticket is drawn randomly from the box, what is the probability that it is red ***or*** yellow?

Three of the five tickets in the box are either red or yellow so

$$P(\text{red or yellow}) = 3/5 = 1/5 + 2/5 = P(\text{red}) + P(\text{yellow}).$$

Example 11. Two tickets are drawn at random, with replacement from the box above. What is the probability that the first ticket is red ***or*** the second ticket is yellow?

It is tempting to add the probabilities, as we did above:

$$\begin{aligned} P((\text{first red}) \text{ or } (\text{second yellow})) &= P(\text{first red}) + P(\text{second yellow}) \\ &= 1/5 + 2/5 \\ &= 3/5 = 60\%. \end{aligned}$$

But this would be wrong in this case...

The table below lists all the possible pairs of tickets that we can draw (with replacement) from our box of five tickets. In this table, $y1$ and $y2$ are the first and second yellow tickets in the box, and $b1$ and $b2$ are the first and second blue tickets.

(r,r)	(r,b1)	(r,b2)	(r,y1)	(r,y2)
(b1,r)	(b1,b1)	(b1,b2)	(b1,y1)	(b1,y2)
(b2,r)	(b2,b1)	(b2,b2)	(b2,y1)	(b2,y2)
(y1,r)	(y1,b1)	(y1,b2)	(y1,y1)	(y1,y2)
(y2,r)	(y2,b1)	(y2,b2)	(y2,y1)	(y2,y2)

($\star$) The outcomes listed above are all equally likely (why?).

($\star$) The first *row* of the table includes all pairs where the *first* ticket is red.

($\star$) The last two *columns* include all pairs where the *second* ticket is yellow.

($\star$) 13 of the 25 pairs of draws have the feature we want (first-ticket-red ***or*** second-ticket-yellow), so

$$P((\text{first-ticket-red}) \text{ or } (\text{second-ticket-yellow})) = 13/25 = 52\%.$$

What is the difference between the two examples...?

The two events in the first example are ***mutually exclusive***: if the (single) ticket we draw is red, then it cannot be yellow and vice versa.

The two events in the second example are ***not mutually exclusive***. It is possible that the first ticket will be red *and* the second ticket will be yellow. In fact the chance that this happens is $2/25 = 8\%$.

Simply adding the chances in the second example, as in the first example, overestimates the probability. The pairs *first-ticket-red* **and** *second-ticket-yellow* are counted twice, and the chance of these pairs should be subtracted from the sum of the probabilities to get the right answer:

$$P((\text{first-red}) \ \textbf{or} \ (\text{second-yellow})) = 1/5 + 2/5 - 2/25 = 13/25 = 52\%.$$

Simple rule to remember: *If the events E and F are mutually exclusive, which means that $P(E \text{ and } F) = 0$, then*

$$P(E \text{ or } F) = P(E) + P(F).$$

Example 12. Suppose that a fair coin is tossed 4 times...

(*) In this context, *tossing a fair coin* is taken to mean that

$$P(\text{heads}) = P(\text{tails}) = 50\%$$

on each toss and also that the results of different tosses are *independent* of each other.

(i) *What are the chances that we observe* 0 *heads in* 4 *tosses?*

Observing 0 heads means that each of the four tosses resulted in tails, so

$$\begin{aligned} P(0 \text{ H in 4 tosses}) &= P(4 \text{ Ts in 4 tosses}) \\ &= P(\text{T on 1st and T on 2nd and T on 3rd and T on 4th}) \\ &= P(T) \cdot P(T) \cdot P(T) \cdot P(T) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16} = 6.25\% \end{aligned}$$

(ii) *What is the probability of observing exactly* 1 *head in* 4 *tosses?*

To observe exactly 1 head in 4 tosses means that we observe one of the 4 sequences

$$HTTT,\ THTT,\ TTHT,\ TTTH.$$

In other words, "observing 1 head in 4 tosses" is the same as "observing $HTTT$ <u>or</u> $THTT$ <u>or</u> $TTHT$ <u>or</u> $TTTH$."

These four sequences are all mutually exclusive of each other (seeing one of them excludes the possibility that you saw one of the others), so

$$\begin{aligned} P(1 \text{ H in 4 tosses}) &= P(HTTT) + P(THTT) + P(TTHT) + P(TTTH) \\ &= P(H)P(T)P(T)P(T) + P(H)P(T)P(T)P(T) \\ &\qquad + P(T)P(T)P(H)P(T) + P(T)P(T)P(T)P(H) \\ &= 0.0625\% + 0.0625\% + 0.0625\% = 0.0625\% = 25\% \end{aligned}$$

because

$$P(HTTT) = P(THTT) = P(TTHT) = P(TTTH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 6.25\%$$

(iii) *What is the probability of observing exactly* 3 *heads in* 4 *tosses?*

Observing 3 heads in 4 tosses means observing 1 tail in 4 tosses. Since the chances for heads and tails are the same, the probability of observing exactly 1 tail in 4 tosses is the same as the probability of observing 1 head in 4 tosses. We already figured this out...

$$P(3 \text{ H in } 4 \text{ tosses}) = P(1 \text{ T in } 4 \text{ tosses}) = P(1 \text{ H in } 4 \text{ tosses}) = 25\%$$

(iv) *What is the probability of observing exactly* 4 *heads in* 4 *tosses?*

Observing 4 heads in 4 tosses is the same as observing 0 tails in 4 tosses, and this has the same probability as observing 0 heads in 4 tosses...

$$P(4 \text{ H in } 4 \text{ tosses}) = P(0 \text{ T in } 4 \text{ tosses}) = P(0 \text{ H in } 4 \text{ tosses}) = 6.25\%$$

You may have noticed that I skipped the question about 2 heads in 4 tosses...

(v) *What is the probability of observing* 2 *heads in* 4 *tosses?*

When you toss a coin 4 times, you observe either 0 heads, 1 head, 2 heads, 3 heads or 4 heads. Moreover, these events are all mutually exclusive (you can't observe both 3 heads and 0 heads, for example.

This means that the event $E =$'not 2 heads in 4 tosses' is the same as the event '0 heads, 1 head, 3 heads or 4 heads in 4 tosses'. So we can find the probability of 2 heads in 4 tosses, using the rule for *complements*...

$$
\begin{aligned}
P(\text{2 heads in 4 tosses}) &= 1 - P(\text{not 2 heads in 4 tosses}) \\
&= 1 - P(\text{0 heads or 1 head or 3 heads or 4 heads}) \\
&= 1 - (P(\text{0 heads}) + P(\text{1 head}) + P(\text{3 heads}) + P(\text{4 heads})) \\
&\text{(because the events are mutually exclusive)} \\
&= 1 - (0.0625 + 0.25 + 0.25 + 0.0625) = 1 - 0.625 = 0.375
\end{aligned}
$$

The probabilities for each of the five number of heads in 4 tosses of a fair coin are displayed on the next slide in a *probability histogram*

Probability histogram for the possible number of heads in 4 tosses of a fair coin: the area (and height) of each bar is equal to the probability of observing the number of heads over which the bar is centered.